Nullity distributions associated to Cartan connection*

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Abstract

The Klein-Grifone approach to global Finsler geometry is adopted. The nullity distributions of the three curvature tensors of Cartan connection are investigated. Nullity distributions concerning certain relevant special Finsler spaces are considered. Concrete examples are given whenever the situation needs.

Keywords: Barthel connection, Cartan Connection, nullity distribution, nullity foliation, index of nullity, completely integrable, auto-parallel, Berwald space, Landesberg space, h-isotropic space, S_3-like space.

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1. Introduction

Chern and Kuiper [6] in 1952 defined a distribution on a Riemannian manifold M which assigns to each point $x \in M$, the subspace

$$\mathcal{N}_R(x) = \{X \in T_xM : R(X,Y) = 0, \forall Y \in T_xM\},$$

where R is the curvature of the Riemannian connection on M. It is called the nullity space at x. The distribution defined by the subspace $\mathcal{N}_x$ at each point x of M is called the nullity distribution $\mathcal{N}$ of the Riemannian manifold M. The dimension μ_x of $\mathcal{N}_x$ is called the index of nullity at x. Chern and Kuiper showed that if μ_x is constant in a neighborhood then $\mathcal{N}$ constitutes a completely integrable distribution there, and that the leaves of the resulting foliation are flat. Later, Maltz [12] showed that the leaves are also totally geodesic.

In 1972, Akbar Zadeh [2], [3] extended this work to Finsler geometry adopting the **pullback approach** to global Finsler geometry. He studied the nullity distribution of the (classical) curvature of Cartan connection. Recently, Bidabad and Refie-Rad [4] studied a more general case called k-nullity distribution in Finsler geometry. On the other hand, in 1982, Youssef [14], [15] studied the nullity distributions of the curvature tensors of Barthel connection and Berwald connection, adopting the **Klein-Grifone approach** to global Finsler geometry.

In the present paper, we investigate the nullity distribution of the three curvature tensors of Cartan connection adopting the **Klein-Grifone approach** [8], [9], and [10]. The paper is organized as follows. In the first section, we give the necessary material that will be needed throughout the present work. In particular, we give a brief account on the Klein-Grifone approach to global Finsler geometry. In the second section, we focus our attention on the most important properties and formulas related to the curvature tensors of Cartan connection. In the third section, we investigate the nullity distribution (ND) $\mathcal{N}_R$ of the h-curvature tensor R of Cartan connection, the nullity spaces being subspaces of the horizontal space. We show that the ND $\mathcal{N}_R$ is included in the ND of the curvature of Barthel connection and we show, by an example, that this inclusion is proper. We show that the ND $\mathcal{N}_R$ is completely integrable and the leaves of the nullity foliation are auto-parallel and hence totally geodesic submanifolds. In the Fourth and fifth sections, we study the ND's of the hv-curvature and v-curvature of Cartan connection. We show through examples that these ND's are not completely integrable. Nevertheless, we investigate necessary and sufficient conditions for such distributions to be completely integrable.

It should be noted that in the pullback approach ([2], [3]) the ND of the classical curvature of Cartan connection is completely integrable and, consequently, the ND's of the h-curvature, hv-curvature and v-curvature are completely integrable. However, in the Klein-Grifone approach the situation is different: the ND of the h-curvature is completely integrable whereas the ND's of the hv-curvature and v-curvature are not.

Throughout the paper, we give concrete examples whenever the situation needs. Moreover, we study ND's related to certain special Finsler spaces relevant to the situation under consideration.

2. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the Klein-Grifone approach to global Finsler geometry. For more details, we refer to [8], [9], and [10]. We make the assumption that the geometric objects we consider are of class C^∞.
The following notations will be used throughout this paper:
M: a real differentiable manifold of finite dimension n and of class C^∞,
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \longrightarrow M$: the tangent bundle of M,
$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,
df : the exterior derivative of f,
$d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L,
$\mathcal{L}_X$: the Lie derivative with respect to $X \in \mathfrak{X}(M)$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(TM)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. The vertical vector field C on TM defined by $C := \gamma \circ \overline{\eta}$, where $\overline{\eta}$ is the vector field on $\pi^{-1}(TM)$ given by $\overline{\eta}(u) = (u, u)$, is called the fundamental or the canonical (Liouville) vector field.

In this work, we shall need the evaluation of the Frölicher-Nijenhuis bracket in some special cases [7]:

If L is a vector ℓ-form and $X \in \mathfrak{X}(M)$, then, for all $Y_1, ..., Y_\ell \in \mathfrak{X}(M)$,

$$[X, L](Y_1, ..., Y_\ell) = [X, L(Y_1, ..., Y_\ell)] - \sum_{i=1}^{\ell} L(Y_1, ..., [X, Y_i], ..., Y_\ell).$$

In particular, if L is vector 1-form,

$$[X, L]Y = [X, LY] - L[X, Y].$$

If K and L are vector 1- forms, then

$$\begin{aligned}[K, L](X, Y) = {} & [KX, LY] + [LX, KY] + KL[X, Y] + LK[X, Y] \\ & -K[LX, Y] - K[X, LY] - L[KX, Y] - L[X, KY].\end{aligned}$$

In particular, the vector 2-form $N_K := \frac{1}{2}[K, K]$ is said to be the Nijenhuis torsion of the vector 1-form K:

$$N_K := \frac{1}{2}[K, K](X, Y) = [KX, KY] + K^2[X, Y] - K[KX, Y] - K[X, KY]. \quad (2.1)$$

One can show that the natural almost tangent structure J has the properties:

$$J^2 = 0, \quad [J, J] = 0, \quad [C, J] = -J, \quad \mathrm{Im}(J) = Ker(J) = V(TM), \tag{2.2}$$

A scalar p-form ω on TM is semi-basic if $i_{JX}\omega = 0$, $\forall X \in \mathfrak{X}(TM)$. A vector ℓ-form L on TM is semi-basic if $JL = 0$ and $i_{JX}L = 0$, $\forall X \in \mathfrak{X}(TM)$.

A scalar p-form ω on TM is homogenous of degree r if $\mathcal{L}_C\omega = r\omega$. A vector ℓ-form L on TM is homogenous of degree r, denoted h(r), if $[C, L] = (r-1)L$. It is clear that J is h(0).

A semispray on M is a vector field S on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $JS = C$. A semispray S which is homogeneous of degree 2 ($[C, S] = S$) is called a spray.

A nonlinear connection on M is a vector 1-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

The vertical and horizontal projectors v and h associated with Γ are defined respectively by $v := \frac{1}{2}(I - \Gamma)$, $h := \frac{1}{2}(I + \Gamma)$. Thus Γ gives rise to the direct sum decomposition $TTM = V(TM) \oplus H(TM)$, where $V(TM) := Im\, v = Ker\, h$ is the vertical bundle and $H(TM) := Im\, h = \ker v$ is the horizontal bundle induced by Γ. An element of $V(TM)$ (resp. $H(TM)$) will be denoted by vX (resp. hX). We have $Jv = 0$, $vJ = J$, $Jh = J$, $hJ = 0$. A nonlinear connection Γ is homogeneous if $[C, \Gamma] = 0$. To each nonlinear connection Γ, one can associate a semispray S which is horizontal with respect to Γ, namely, $S = hS'$, where S' is an arbitrary semispray. Moreover, if Γ is homogeneous, then its associated semispray is a spray.

The torsion t of a nonlinear connection Γ is the vector 2-form on TM defined by $t := \frac{1}{2}[J, \Gamma]$. The curvature of Γ is the vector 2-form on TM defined by $\mathfrak{R} := -\frac{1}{2}[h, h]$. Associated with Γ, an almost complex structure F ($F^2 = -I$) is defined by $FJ = h$ and $Fh = -J$. This F defines an isomorphism of T_zTM for all $z \in TM$.

Definition 2.1. [10] *A Finsler space of dimension n is a pair (M, E), where M is a differentiable manifold of dimension n and E is a map*

$$E : TM \longrightarrow \mathbb{R},$$

called the energy function, satisfying the axioms:

(a) *$E(u) > 0$ for all $u \in \mathcal{T}M$ and $E(0) = 0$,*

(b) *E is C^∞ on $\mathcal{T}M$, C^1 on TM,*

(c) *E is homogenous of degree 2: $\mathcal{L}_C E = 2E$,*

(d) *The exterior 2-form $\Omega := dd_J E$, called the fundamental form, has maximal rank.*

Theorem 2.2. [10] *Let (M, E) be a Finsler space. The vector field $S \in \mathfrak{X}(TM)$ defined by $i_S\Omega = -dE$ is a spray. Such a spray is called the canonical spray associated with (M, E).*

Now, we give a fundamental result which ensures the existence and uniqueness of a remarkable nonlinear connection.

Theorem 2.3. [10] *On a Finsler space (M, E), there exists a unique conservative $(d_h E = 0)$ homogeneous nonlinear connection with zero torsion. It is given by:*

$$\Gamma = [J, S],$$

where S is the canonical spray. Such a connection is called the canonical connection, Barthel connection or Cartan nonlinear connection associated with (M, E).

It should be noted that the semi-spray associated with the Barthel connection is a spray, which is the canonical spray.

3. Berwald and Cartan connections

In this section, we present the necessary material, concerning Berwald and Cartan connections, that will be needed throughout the present work. For more details, we refer to [9] and [15].

Theorem 3.1. *[9] For a Finsler space (M, E), there exists a unique linear connection $\overset{\circ}{D}$ on TM satisfying the following properties:*

(a) $\overset{\circ}{D} J = 0$.

(b) $\overset{\circ}{D} C = v$.

(c) $\overset{\circ}{D} \Gamma = 0$ $(\Longleftrightarrow \overset{\circ}{D} h = \overset{\circ}{D} v = 0)$.

(d) $\overset{\circ}{D}_{JX} JY = J[JX, Y]$.

(e) $\overset{\circ}{T}(JX, Y) = 0$,

where h and v are the horizontal and vertical projectors of Barthel connection and $\overset{\circ}{T}$ is the (classicl) torsion of $\overset{\circ}{D}$. This connection is called the Berwald connection.

The explicit expression of $\overset{\circ}{D}$ is given by:

$$\left.\begin{aligned} \overset{\circ}{D}_{JX} JY &= J[JX, Y], \\ \overset{\circ}{D}_{hX} JY &= v[hX, JY], \\ \overset{\circ}{D} F &= 0. \end{aligned}\right\} \tag{3.1}$$

Lemma 3.2. *The Berwald connection has the property that*

$$\overset{\circ}{T}(hX, hY) = \mathfrak{R}(X, Y),$$

where $\mathfrak{R}$ is the curvature of Barthel connection.

Let (M, E) be a Finsler space and $\Omega := dd_J E$. The map $\overline{g}$ defined by

$$\overline{g}(JX, JY) := \Omega(JX, Y), \ \forall \ X, Y \in T(TM)$$

defines a metric on $V(TM)$. This metric can be extended to a metric g on $T(TM)$ defined by the formula:

$$g(X, Y) = \overline{g}(JX, JY) + \overline{g}(vX, vY) = \Omega(X, FY). \tag{3.2}$$

Theorem 3.3. *[9] For a Finsler space (M, E), there exists a unique linear connection D on TM satisfying the following properties:*

(a) $DJ = 0$. **(b)** $DC = v$.

(c) $D\Gamma = 0$ ($\Longleftrightarrow Dh = Dv = 0$). **(d)** $Dg = 0$.

(e) $T(JX, JY) = 0$. **(f)** $JT(hX, hY) = 0$.

This connection is called the Cartan connection.

The explicit expression of D is given by:

$$\left.\begin{aligned} D_{JX}JY =& \overset{\circ}{D}_{JX}JY + \mathcal{C}(X,Y), \\ D_{hX}JY =& \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X,Y), \\ DF =& \, 0, \end{aligned}\right\} \tag{3.3}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are the scalar 2-forms on TM defined by

$$\Omega(\mathcal{C}(X,Y), Z) = \frac{1}{2}(\mathcal{L}_{JX}(J^*g))(Y,Z), \qquad \Omega(\mathcal{C}'(X,Y), Z) = \frac{1}{2}(\mathcal{L}_{hX}g)(JY,JZ),$$

with $(J^*g)(Y,Z) = g(JY,JZ)$.

The tensors $\mathcal{C}$ and $\mathcal{C}'$ will be called the first and second Cartan tensors respectively. They are semi-basics, symmetric and

$$\mathcal{C}(X,S) = \mathcal{C}'(X,S) = 0. \tag{3.4}$$

We have the following lemmas.

Lemma 3.4. *The (h)h-torsion $T(hX, hY)$ and (h)v-torsion $T(hX, JY)$ of Cartan connection are given respectively by*

$$T(hX,hY) = \mathfrak{R}(X,Y), \quad T(hX,JY) = (\mathcal{C}' - F\mathcal{C})(X,Y),$$

where $\mathfrak{R}$ is the curvature of Barthel connection.

Lemma 3.5. *The h-curvature R, hv-curvature P and v-curvature Q of Cartan connection are given respectively by:*

(a) $$\begin{aligned} R(X,Y)Z =& \overset{\circ}{R}(X,Y)Z + (D_{hX}\mathcal{C}')(Y,Z) - (D_{hY}\mathcal{C}')(X,Z) + \mathcal{C}'(F\mathcal{C}'(X,Z),Y) \\ & - \mathcal{C}'(F\mathcal{C}'(Y,Z),X) + \mathcal{C}(F\mathfrak{R}(X,Y),Z). \end{aligned}$$

(b) $$\begin{aligned} P(X,Y)Z =& \overset{\circ}{P}(X,Y)Z + (D_{hX}\mathcal{C})(Y,Z) - (D_{JY}\mathcal{C}')(X,Z) + \mathcal{C}(F\mathcal{C}'(X,Z),Y) \\ & + \mathcal{C}(F\mathcal{C}'(X,Y),Z) - \mathcal{C}'(F\mathcal{C}(Y,Z),X) - \mathcal{C}'(F\mathcal{C}(X,Y),Z). \end{aligned}$$

(c) $Q(X,Y)Z = \mathcal{C}(F\mathcal{C}(X,Z),Y) - \mathcal{C}(F\mathcal{C}(Y,Z),X)$,

where $\overset{\circ}{R}$ and $\overset{\circ}{P}$ are respectively the h-curvature and hv-curvature of Berwald connection.

Lemma 3.6. *For Cartan connection, the following properties hold:*

(a) $R(X,Y)S = \mathfrak{R}(X,Y)$.

(b) $P(X,Y)S = \mathcal{C}'(X,Y)$.

(c) $P(S,X)Y = P(X,S)Y = 0$.

(d) $Q(S,X)Y = Q(X,S)Y = Q(X,Y)S = 0$.

Lemma 3.7. *The Bainchi identities for Cartan connection are given by:*

(a) $\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}(F\mathfrak{R}(X,Y),Z)\}$.

(b) $\mathfrak{S}_{X,Y,Z}\{Q(X,Y)Z\} = 0$.

(c) $\mathcal{C}(F\mathfrak{R}(X,Y),Z) = \mathfrak{R}(F\mathcal{C}(X,Z),Y) - \mathfrak{R}(F\mathcal{C}(Y,Z),X)$.

(d) $\mathfrak{S}_{X,Y,Z}\{(D_{hX}\mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}'(F\mathfrak{R}(X,Y),Z)\}$.

(e) $\mathfrak{S}_{X,Y,Z}\,\{(D_{hX}R)(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\,\{P(X,F\mathfrak{R}(Y,Z))\}$.

(f) $(D_{hX}P)(Y,Z) - (D_{hY}P)(X,Z) + (D_{JZ}R)(X,Y) = P(X,F\mathcal{C}'(Y,Z))$
$- P(Y,F\mathcal{C}'(X,Z)) + R(F\mathcal{C}(Y,Z),X) - R(F\mathcal{C}(X,Z),Y) - Q(F\mathfrak{R}(X,Y),Z)$.

(g) $(D_{hX}Q)(Y,Z) - (D_{JY}P)(X,Z) + (D_{JZ}P)(X,Y) = P(F\mathcal{C}(X,Y),Z)$
$- P(F\mathcal{C}(Z,X),Y) - Q(F\mathcal{C}'(X,Y),Z) + Q(F\mathcal{C}'(Z,X),Y)$.

(h) $\mathfrak{S}_{X,Y,Z}\{(D_{JX}Q)(Y,Z)\} = 0$,

where $\mathfrak{S}_{X,Y,Z}$ *is the cyclic sum over the vector fields* X, Y *and* Z.

4. Nullity distribution of Cartan h-curvature

We are now in a position to study the nullity distributions associated to Cartan connection. Firstly, we study the nullity distribution of the h-curvature tensor. It should be noted that the nullity distributions of Barthel and Berwald connections have been investigated in [14] and [15].

We need the following lemma for subsequent use.

Lemma 4.1. *For all* $X, Y \in \mathfrak{X}(TM)$, *we have*

(a) $[JX, JY] = J(D_{JX}Y - D_{JY}X)$.

(b) $[hX, JY] = J(D_{hX}Y) - h(D_{JY}X) - (\mathcal{C}' - F\mathcal{C})(X,Y)$.

(c) $[hX, hY] = h(D_{hX}Y - D_{hY}X) - \mathfrak{R}(X,Y)$.

Proof.
(a) Using (3.1) and (3.3), by the symmetry of $\mathcal{C}$ and since $[J, J] = 0$, $J^2 = 0$ and $DJ = 0$, we get

$$\begin{aligned} J(D_{JX}Y - D_{JY}X) &= D_{JX}JY - D_{JY}JX \\ &= \overset{\circ}{D}_{JX}JY + \mathcal{C}(X, Y) - \overset{\circ}{D}_{JY}JX - \mathcal{C}(Y, X) \\ &= J[JX, Y] - J[JY, X] \\ &= [JX, JY]. \end{aligned}$$

(b) Using (3.1) and (3.3), by the symmetry of $\mathcal{C}$ and since $DJ = Dh = DF = 0$, we obtain

$$\begin{aligned} J(D_{hX}Y) - h(D_{JY}X) &= D_{hX}JY - D_{JY}hX \\ &= \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X, Y) - \overset{\circ}{D}_{JY}hX - F\mathcal{C}(Y, X) \\ &= v[hX, JY] - h[JY, X] + (\mathcal{C}' - F\mathcal{C})(X, Y) \\ &= [hX, JY] + (\mathcal{C}' - F\mathcal{C})(X, Y). \end{aligned}$$

(c) Again using (3.1) and (3.3), by the symmetry property of $\mathcal{C}'$, we have

$$\begin{aligned} h(D_{hX}Y - D_{hY}X) &= D_{hX}hY - D_{hY}hX \\ &= \overset{\circ}{D}_{hX}hY + F\mathcal{C}'(X, Y) - \overset{\circ}{D}_{hY}hX - F\mathcal{C}'(Y, X) \\ &= Fv[hX, JY] + Fv[JX, hY]. \end{aligned}$$

As the torsion of Γ vanishes, then $0 = t(X, Y) = v[JX, hY] + v[hX, JY] - J[hX, hY]$, from which $Fv[JX, hY] + Fv[hX, JY] = FJ[hX, hY] = h[hX, hY]$. Consequently,

$$h(D_{hX}Y - D_{hY}X) = h[hX, hY] = [hX, hY] - v[hX, hY] = [hX, hY] + \mathfrak{R}(X, Y),$$

where we have used the identity $\mathfrak{R}(X, Y) = -v[hX, hY]$ [14]. □

Remark 4.2. It is to be noted that the identity $\mathfrak{R}(X, Y) = -v[hX, hY]$ shows that the Lie bracket of two horizontal vector fields is horizontal if and only if the curvature $\mathfrak{R}$ vanishes. This means that a necessary and sufficient condition for the horizontal distribution to be completely integrable is that $\mathfrak{R}$ vanishes. This fact can also be deduced from Lemma 4.1 **(c)** above.

Definition 4.3. *Let R be the h-curvature tensor of Cartan connection. The nullity space of R at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by*

$$\mathcal{N}_R(z) := \{X \in H_z(TM) : R(X, Y) = 0, \ \forall Y \in T_z(TM)\}.$$

The dimension of $\mathcal{N}_R(z)$, denoted by $\mu_R(z)$, is the index of nullity of R at z.

If the index of nullity is constant, then the map $\mathcal{N}_R : z \mapsto \mathcal{N}_R(z)$ defines a distribution $\mathcal{N}_R$ of dimension μ_R called nullity distribution of R.

Any vector field belonging to the nullity distribution is called a nullity vector field.

Proposition 4.4. *The nullity distribution $\mathcal{N}_R$ has the following properties:*

(a) $\mathcal{N}_R \neq \phi$.

(b) $\mathcal{N}_R \subseteq \mathcal{N}_{\mathfrak{R}}$, *where* $\mathcal{N}_{\mathfrak{R}}$ *is the nullity distribution of the curvature* $\mathfrak{R}$.

(c) *If* $Z \in \mathcal{N}_R$, *then* $R(X,Y)Z = \mathcal{C}(F\mathfrak{R}(X,Y), Z)$.

(d) *If* $S \in \mathcal{N}_R$, *then* $\mathfrak{R} = 0$.

(e) *If* $X \in \mathcal{N}_R$, *then* $[C, X] \in \mathcal{N}_R$ *and consequently,* $[C, X] \in \mathcal{N}_{\mathfrak{R}}$.

Proof.
(b) Let X be a nullity vector field. We have

$$\begin{aligned} X \in \mathcal{N}_R &\Longrightarrow R(X,Y)Z = 0 \quad \forall Y, Z \in \mathfrak{X}(TM) \\ &\Longrightarrow R(X,Y)S = 0 \quad \forall Y \in \mathfrak{X}(TM) \\ &\Longrightarrow \mathfrak{R}(X,Y) = 0 \qquad \forall Y \in \mathfrak{X}(TM) \\ &\Longrightarrow X \in \mathcal{N}_{\mathfrak{R}}. \end{aligned}$$

(c) Let $Z \in \mathcal{N}_R$, then $Z \in \mathcal{N}_{\mathfrak{R}}$ and by Lemma 3.7 **(a)**, we have

$$\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}(F\mathfrak{R}(X,Y), Z)\}.$$

Since $R(Y,Z)X = R(Z,X)Y = 0$ and $\mathfrak{R}(Y,Z) = \mathfrak{R}(Z,X) = 0$, the result follows.

(d) Let $S \in \mathcal{N}_R$, then by **(c)**, we have $R(X,Y)S = \mathcal{C}(F\mathfrak{R}(X,Y), S)$. Then, the result follows from (3.4) and Lemma 3.6.

(e) Let $X \in \mathcal{N}_R$. By the identity $D_C R = 0$ [9], we have

$$(D_C R)(X,Y) = 0,$$

which leads to

$$R(D_C X, Y) = 0.$$

Using (3.1) and (3.3), we have $R([C,X],Y) = 0$. Since h is h(1), then $[C,h] = 0$, from which $[C,hX] = h[C,X]$. That is, $[C,hX]$ is horizontal. Hence, $[C,X] \in \mathcal{N}_R$. Consequently, by **(b)**, $[C,X] \in \mathcal{N}_{\mathfrak{R}}$. □

It is important to note that the converse of property **(b)** of Proposition 4.4 is not true in general, that is, $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R$. This is shown by the next example in which the calculations are performed using MAPLE program.

Example 4.5. Let $M = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \neq 0\}$,
$U = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x_4 \neq 0;\ y_i \neq 0, i = 1, ..., 4\} \subset TM$.
Let the energy function E be defined on the open subset U of TM by:
$E = x_4 y_1 (y_2^3 + y_3^3 + y_4^3)^{1/3}$. Then, we have:

$$\Omega = \frac{1}{2(y_2^3 + y_3^3 + y_4^3)^{2/3}}\{-(y_2^3 + y_3^3 + y_4^3)\, dx_1 \wedge dx_4 - y_1 y_2^2\, dx_2 \wedge dx_4 - y_1 y_3^2\, dx_3 \wedge dx_4$$
$$-2x_4 y_2^2\,(dx_1 \wedge dy_2 + dx_2 \wedge dy_1) - 2x_4 y_3^2\,(dx_1 \wedge dy_3 + dx_3 \wedge dy_1)$$

$$-2x_4y_4^2\,(dx_1\wedge dy_4+dx_4\wedge dy_1)\}-\frac{2x_4}{(y_2^3+y_3^3+y_4^3)^{5/3}}\{y_1y_2(y_3^3+y_4^3)\,dx_2\wedge dy_2$$
$$-y_1y_2^2y_3^2\,(dx_2\wedge dy_3+dx_3\wedge dy_2)-y_1y_2^2y_4^2\,(dx_2\wedge dy_4+dx_2\wedge dy_4)$$
$$+y_1y_3(y_2^3+y_4^3)\,dx_3\wedge dy_3-y_1y_3^2y_4^2\,(dx_3\wedge dy_4+dx_4\wedge dy_3)$$
$$+y_1y_4(y_3^3+y_4^3)\,dx_4\wedge dy_4\}.$$

The identity $i_S\Omega=-dE$ gives the following non-vanishing coefficients of the canonical spray S^i:

$$S^2=\frac{3y_2y_4}{4x_4},\qquad S^3=\frac{3y_3y_4}{4x_4},\qquad S^4=-\frac{y_2^3+y_3^3-2y_4^3}{4x_4y_4}.$$

The non-vanishing coefficients of Barthel connection Γ^i_j are:

$$\Gamma^2_2=\frac{3y_4}{4x_4},\qquad \Gamma^2_4=\frac{3y_2}{4x_4},\qquad \Gamma^3_3=\frac{3y_4}{4x_4},\qquad \Gamma^3_4=\frac{3y_3}{4x_4},$$

$$\Gamma^4_2=-\frac{3y_2^2}{4x_4y_4},\qquad \Gamma^4_3=-\frac{3y_3^2}{4x_4y_4},\qquad \Gamma^4_4=\frac{y_2^3+y_3^2+4y_4^3}{4x_4y_4^2}.$$

The independent non-vanishing components of the curvature $\mathfrak{R}^i_{jk}$ of Barthel connection are:

$$\mathfrak{R}^2_{23}=\frac{9y_3^2}{16x_4^2y_4},\qquad \mathfrak{R}^2_{24}=-\frac{3(y_2^3+y_3^2+5y_4^3)}{16x_4^2y_4^2},$$

$$\mathfrak{R}^3_{23}=-\frac{9y_2^2}{16x_4^2y_4},\qquad \mathfrak{R}^3_{34}=-\frac{3(y_2^3+y_3^2+5y_4^3)}{16x_4^2y_4^2},$$

$$\mathfrak{R}^4_{24}=\frac{3y_2^2(y_2^3+y_3^2+5y_4^3)}{16x_4^2y_4^4},\qquad \mathfrak{R}^4_{34}=\frac{3y_3^2(y_2^3+y_3^2+5y_4^3)}{16x_4^2y_4^4}.$$

Now, let $X\in\mathcal{N}_{\mathfrak{R}}$, then X can be written in the form $X=X^1h_1+X^2h_2+X^3h_3+X^4h_4$, where X^1,X^2,X^3,X^4 are the components of the nullity vector X with respect to the basis $\{h_1,h_2,h_3,h_4\}$ of the horizontal space, where $h_i:=\frac{\partial}{\partial x^i}-\Gamma^m_i\frac{\partial}{\partial y^m}$, $i,m=1,...,4$. The equation $\mathfrak{R}(X,Y)=0$, $\forall\,Y\in H(TM)$, is written locally in the form

$$X^j\mathfrak{R}^i_{jk}=0.$$

This is equivalent to the system of equations:

$$3y_3^2X^3-(y_2^3+y_3^3+5y_4^3)X^4=0,$$

$$y_3^2X^2=0,$$

$$y_2^2X^3=0.$$

From the above system, we have $X^1=t_1$, $t_1\in\mathbb{R}$ and $X^2=X^3=0$. Then, we get $(y_2^3+y_3^3+5y_4^3)X^4=0$. Now, we have two cases, either $y_2^3+y_3^3+5y_4^3=0$ or $y_2^3+y_3^3+5y_4^3\neq 0$. Firstly, if $y_2^3+y_3^3+5y_4^3\neq 0$, then $X^4=0$ and thus $\mu_{\mathfrak{R}}=1$. Secondly, if $y_2^3+y_3^3+5y_4^3=0$, then $X^4=t_4$, $t_4\in\mathbb{R}$ and thus $X=t_1h_1+t_4h_4$ and $\mu_{\mathfrak{R}}=2$. We will be interested in the second case.

Calculations using MAPLE give the coefficients of Cartan connection Γ^i_{jk} and so the components of the h-curvature tensor R^h_{ijk}. Taking into account that $y_2^3 + y_3^3 + 5y_4^3 = 0$, the independent non-vanishing components R^h_{ijk} are as follows:

$$R^2_{123} = \frac{-9y_3^2}{32x_4^2 y_1 y_4}, \qquad R^3_{123} = \frac{9y_2^2}{32x_4^2 y_1 y_4}, \qquad R^1_{223} = \frac{-9y_1 y_2 y_3^2}{64x_4^2 y_4^4}$$

$$R^2_{223} = \frac{-9y_2^2 y_3^2}{128x_4^2 y_4^4}, \qquad R^3_{223} = \frac{-9y_2(4y_2^3 + 6y_3^3)}{256x_4^2 y_4^4}, \qquad R^4_{223} = \frac{-45y_2 y_3^2}{128x_4^2 y_4^3},$$

$$R^3_{224} = \frac{-108y_2 y_3}{256x_4^2 y_4^2}, \qquad R^4_{224} = \frac{-3y_2(y_3^3(2y_3^3 - 30y_4^3) - y_2^3(2y_2^3 + 14y_4^3) - 20y_4^6)}{256x_4^2 y_4^7},$$

$$R^3_{234} = \frac{27y_2^2}{64x_4^2 y_4^2}, \qquad R^4_{234} = \frac{27y_2^2 y_3^2}{64x_4^2 y_4^4}, \qquad R^1_{323} = \frac{9y_1 y_2^2 y_3}{64x_4^2 y_4^4}, \qquad R^2_{323} = \frac{9y_3(y_2^3 - 8y_4^3)}{128x_4^2 y_4^4},$$

$$R^3_{323} = \frac{9y_2^2 y_3^2}{128x_4^2 y_4^4}, \qquad R^4_{323} = \frac{45y_2^2 y_3}{128x_4^2 y_4^3}, \qquad R^2_{324} = \frac{27y_3^2 y_4^3}{64x_4^2 y_4^5}, \qquad R^4_{324} = \frac{-27y_2^2 y_3^2}{64x_4^2 y_4^4},$$

$$R^2_{334} = \frac{-27y_2 y_3 y_4^3}{64x_4^2 y_4^5}, \qquad R^4_{334} = \frac{3y_3(y_3^3(-3y_2^3 + 4y_4^3) + 5y_4^3(4y_4^3 + 5y_2^3) - 3y_2^6)}{256x_4^2 y_4^7},$$

$$R^3_{423} = \frac{-9y_2^2}{32x_4^2 y_4^2}, \qquad R^3_{424} = \frac{27y_2^2 y_3}{64x_4^2 y_4^3},$$

$$R^2_{424} = \frac{-3(y_3^3(4y_2^3 + 38y_4^3) + 2y_2^3(2y_2^3 + 11y_4^3) + 10y_2^6)}{256x_4^2 y_4^6}, \qquad R^2_{432} = \frac{-9y_3^2}{32x_4^2 y_4^2},$$

$$R^2_{434} = \frac{27y_2 y_3^2}{64x_4^2 y_4^3}, \qquad R^3_{434} = \frac{-34(y_3^6 + 22y_3^3 y_4^3 + y_2^3 y_3^3 + 23y_2^3 y_4^3 - 3y_2^6 + 10y_4^6)}{256x_4^2 y_4^6}.$$

Now, let $X \in \mathcal{N}_R$. The equation $R(X,Y)Z = 0$, $\forall\, Y, Z \in H(TM)$, is written locally in the form

$$X^j R^h_{ijk} = 0.$$

This is equivalent to the system of equations:

$$y_2(5y_2^3 + 7y_3^3 + 9y_4^3)X^3 + 12y_2 y_3 X^4 = 0,$$

$$y_3^2 X^2 = 0,$$

$$y_2^2 X^3 = 0.$$

The above system has the solution $X^1 = t'_1, t'_1 \in \mathbb{R}$ and $X^2 = X^3 = X^4 = 0$. Thus, $X = t'_1 h_1$ and $\mu_R = 1$. So, the dimension of $\mathcal{N}_R = 1$ and the dimension of $\mathcal{N}_{\mathfrak{R}} = 2$, consequently, $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R$. □

Nevertheless, we have some cases in which $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ as the case of Landesberg spaces satisfying certain conditions.

Definition 4.6. *[13] A Finsler space is called Landesberg if the second Cartan tensor vanishes: $\mathcal{C}' = 0$ or, equivalently, if $P = 0$.*

Theorem 4.7. *Let (M, E) be a Landesberg space. If, for all $X \in \mathcal{N}_{\mathfrak{R}}$, $\overset{\circ}{D}_{JZ} X \in \mathcal{N}_{\mathfrak{R}}$, then $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ and hence $\mathcal{N}_{\mathfrak{R}} = \mathcal{N}_R$.*

Proof. Let $(M.E)$ be a Landesberg space. Then, using Lemma 3.5, we get

$$R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + \mathcal{C}(F\mathfrak{R}(X,Y),Z).$$

Let $X \in \mathcal{N}_{\mathfrak{R}}$, by the above equation and the fact that $\overset{\circ}{R}(X,Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X,Y)$ [15], then, $R(X,Y)Z = -\mathfrak{R}(\overset{\circ}{D}_{JZ}X,Y)$. Since $\overset{\circ}{D}_{JZ}X \in \mathcal{N}_{\mathfrak{R}}, \forall X \in \mathcal{N}_{\mathfrak{R}}$, then $R(X,Y)Z = 0$ and then $X \in \mathcal{N}_R$. Consequently, $\mathcal{N}_{\mathfrak{R}} \subset \mathcal{N}_R$ and hence $\mathcal{N}_{\mathfrak{R}} = \mathcal{N}_R$. □

Theorem 4.8. *Let μ_R be constant on an open subset U of TM. Then, the nullity distribution $z \mapsto \mathcal{N}_R(z)$ is completely integrable on U.*

Proof. To prove this theorem we have to show that if $X,Y \in \mathcal{N}_R$, then $[X,Y] \in \mathcal{N}_R$. So, let $X,Y \in \mathcal{N}_R$ and $Z \in H(TM)$. This implies that X and Y are horizontal and $X,Y \in \mathcal{N}_{\mathfrak{R}}$. Then, by Lemma 3.7 **(e)**, we have

$$\mathfrak{S}_{X,Y,Z}\{(D_XR)(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{P(X,F\mathfrak{R}(Y,Z))\}.$$

Since $X,Y \in \mathcal{N}_{\mathfrak{R}}$, then $\mathfrak{R}(X,Y) = \mathfrak{R}(Y,Z) = \mathfrak{R}(Z,X) = 0$. Making use of Lemma 4.1 and the fact that R is semi-basic and $T(hX,hY) = \mathfrak{R}(X,Y)$, we have

$$\begin{aligned}
0 &= \mathfrak{S}_{X,Y,Z}\{(D_XR)(Y,Z)\}\\
&= \mathfrak{S}_{X,Y,Z}\{D_XR(Y,Z) - R(D_XY,Z) - R(Y,D_XZ)\}\\
&= -R(D_XY,Z) - R(Z,D_YX)\\
&= R(D_XY - D_YX,Z)\\
&= R([X,Y] + \mathfrak{R}(X,Y),Z)\\
&= R([X,Y],Z) + R(\mathfrak{R}(X.Y),Z)\\
&= R([X,Y],Z), \quad \forall Z \in H(TM).
\end{aligned}$$

It remains to show that $[X,Y]$ is horizontal. In fact, as $\mathfrak{R}(X,Y) = -v[hX,hY]$ [14], $0 = \mathfrak{R}(X,Y) = -v[X,Y]$, and hence $[X,Y]$ is horizontal. Hence, we have $[X,Y] \in \mathcal{N}_R$. □

Remark 4.9. It should be noted that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection is completely integrable as has been proved in [14].

We have seen that if the index of nullity μ_R is constant, then the nullity distribution $\mathcal{N}_R$ is completely integrable. Then, according to the Frobenius theorem, there exists a foliation of TM by $\mu_R(z)$-dimensional maximal connected submanifolds which are called the leaves, such that $\mathcal{N}_R(z)$ is the tangent space to the leaf at $z \in TM$. In this case we call the foliation induced by the nullity distribution the nullity foliation.

Theorem 4.10. *The leaves of the nullity foliations of $\mathcal{N}_{\mathfrak{R}}$ and $\mathcal{N}_R$ are auto-parallel submanifolds.*

Proof. To prove that $\mathcal{N}_R$ is auto-parallel with respect to Cartan connection, we have to show that if $X,Y \in \mathcal{N}_R$, then $D_XY \in \mathcal{N}_R$.

Let $X,Y \in \mathcal{N}_R$, then $X,Y \in \mathcal{N}_{\mathfrak{R}}$ and $X,Y \in H(TM)$. As $Dh = 0$, then $D_XhY = hD_XY$, i.e., $D_XY \in H(TM)$. By Lemma 3.7 **(e)**, we have

$$\mathfrak{S}_{X,Y,Z}\{(D_XR)(Y,Z)\} = 0.$$

Consequently

$$\mathfrak{S}_{X,Y,Z}\{R(D_XY,Z)\} = 0.$$

Hence $R(D_XY,Z) = 0\ \forall Z \in \mathfrak{X}(TM)$ and $D_XY \in \mathcal{N}_R$.

Similarly, we show that if $X, Y \in \mathcal{N}_{\mathfrak{R}}$, then $D_XY \in \mathcal{N}_{\mathfrak{R}}$. By Lemma 3.7 **(d)**, we have

$$\mathfrak{S}_{X,Y,Z}\{(D_X\mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\mathcal{C}\{(F\mathfrak{R}(X,Y),Z)\}.$$

Since $X, Y \in \mathcal{N}_{\mathfrak{R}}$, then

$$\mathfrak{S}_{X,Y,Z}\{(D_X\mathfrak{R})(Y,Z)\} = 0.$$

Consequently, $\mathfrak{R}(D_XY,Z) = 0\ \forall Z \in \mathfrak{X}(TM)$ and $D_XY \in \mathcal{N}_{\mathfrak{R}}$. □

It is well known that the concepts of auto-parallel submanifold and totally geodesic submanifold coincide in Riemannian geometry [11]. This is not true, in general. However, every auto-parallel submanifold is totally geodesic [5]. So, we have the following corollary.

Corollary 4.11. *The leaves of the nullity foliations $\mathcal{N}_{\mathfrak{R}}$ and $\mathcal{N}_R$ are totally geodesic submanifolds.*

Definition 4.12. *[1], [16] A Finsler space (M,E), where $\dim M \geq 3$, is said to be h-isotropic if there exists a scalar function k_o such that the h-curvature tensor R of Cartan connection has the form*

$$R(X,Y)Z = k_o\{g(X,Z)Y - g(Y,Z)X\},\ \ \forall\, X, Y, Z \in \mathfrak{X}(TM).$$

Theorem 4.13. *For an h-isotropic Finsler space, the index of nullity μ_R takes its maximal value, i.e. $\mu_R = n$.*

Proof. Let X be a non zero nullity vector in $\mathcal{N}_R$ and $Y, Z, W \in \mathfrak{X}(TM)$. Then, by Definition 4.12, we have

$$\begin{aligned} 0 &= k_o\{g(X,Z)Y - g(Y,Z)X\} \\ &= k_o\{g(g(X,Z)Y,W) - g(g(Y,Z)X,W)\} \\ &= k_o\{g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\}. \end{aligned}$$

As g is a metric on TM, its trace is thus $2n$. Taking the trace with respect to the pair Y and W, we get

$$k_o\{2ng(X,Z) - g(X,Z)\} = 0,$$

Again, taking the trace of the above equation, we have

$$2n(2n-1)k_o = 0.$$

which gives $k_o = 0$. Consequently, $R = 0$ and hence $\mu_R = n$. □

Definition 4.14. *[13], [16] A Finsler space (M,E), is said to be Berwald space if the hv-curvature tensor $\overset{\circ}{P}$ of Berwald connection vanishes or, equivalently, $D_{hX}\mathcal{C} = 0$ for all $X \in \mathfrak{X}(TM)$.*

Theorem 4.15. *For a Berwald space, the index of nullity $\mu_{\mathfrak{R}}$ of $\mathcal{N}_{\mathfrak{R}}$ takes its maximal value if and only if the index of nullity μ_R of $\mathcal{N}_R$ takes its maximal value.*

Proof. Let (M, E) be a Berwald space and so $\mathcal{C}' = 0$ [13]. Hence, by Lemma 3.5 **(a)**, the h-curvature of Cartan connection is written in the form

$$R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + \mathcal{C}(F\mathfrak{R}(X,Y), Z). \tag{4.1}$$

Now, let $\mu_{\mathfrak{R}} = n$. Then $\mathfrak{R} = 0$, which is equivalent to $\overset{\circ}{R} = 0$ [15]. Thus, Equation (4.1) yields $R = 0$. Consequently, $\mu_R = n$.

Conversely, let $\mu_R = n$. Hence, by (4.1), $\overset{\circ}{R}(X,Y)Z + \mathcal{C}(Z, F\mathfrak{R}(X,Y)) = 0$. Setting $Z = S$ in this equation, we have $\overset{\circ}{R}(X,Y)S = 0$. But $\overset{\circ}{R}(X,Y)S = \mathfrak{R}(X,Y)$ [15]. Thus, $\mathfrak{R} = 0$, consequently, $\mu_{\mathfrak{R}} = n$. □

5. Nullity distribution of Cartan hv-curvature

In this section, we study the nullity distribution of the hv-curvature of Cartan connection. We show that the nullity distribution $\mathcal{N}_P$ of the hv-curvature P is not completely integrable. We impose a certain condition to make $\mathcal{N}_P$ completely integrable. We present a class of Finsler spaces which guarantees the possibility of such a condition.

Definition 5.1. *Let P be the hv-curvature of Cartan connection. The nullity space of P at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by*

$$\mathcal{N}_P(z) := \{X \in H_z(TM) : P(X,Y) = 0, \ \forall Y \in T_zTM\}.$$

The dimension of $\mathcal{N}_P(z)$, denoted by $\mu_P(z)$, is the index of nullity of P at z.

Proposition 5.2. *The nullity distribution of P has the following properties:*

(a) $\mathcal{N}_P \neq \phi$.

(b) $S \in \mathcal{N}_P$.

(c) *If $X \in \mathcal{N}_P(z)$, then $\mathcal{C}'(X,Y) = 0$, $\forall\, Y \in T_zTM$.*

(d) *If $X, Y \in \mathcal{N}_P \cap \mathcal{N}_{\mathfrak{R}}$, then $R(X,Y)Z = \mathcal{C}'([X,Y], Z)$.*

Proof.
(b) Follows from the fact that $P(S,X)Y = P(X,S)Y = 0$ (Lemma 3.6).

(c) Let $X \in \mathcal{N}_P(z)$,

$$\begin{aligned} X \in \mathcal{N}_P(z) &\Longrightarrow P(X,Y)Z = 0 \quad \forall\, Y, Z \in T_zTM \\ &\Longrightarrow P(X,Y)S = 0 \quad \forall\, Y \in T_zTM \\ &\Longrightarrow \mathcal{C}'(X,Y) = 0 \qquad \forall\, Y \in T_zTM. \end{aligned}$$

(d) Let $X,\ Y \in \mathcal{N}_P \cap \mathcal{N}_{\mathfrak{R}}$. Then, by Proposition 5.2 **(c)**, Lemma 3.5 **(a)** and the identity $\overset{\circ}{R}(X,Y)Z = (\overset{\circ}{D}_{JZ}\mathfrak{R})(X,Y)$ [15], we have

$$R(X,Y)Z = (D_{hX}\mathcal{C}')(Y,Z) - (D_{hY}\mathcal{C}')(X,Z).$$

By Lemma 4.1 **(c)** and the fact that $\mathcal{C}'$ is semi-basic, we get

$$R(X,Y)Z = \mathcal{C}'([hX,hY],Z).$$

Hence, the result follows. □

Theorem 5.3. *For a Landesberg space, the nullity distributions $\mathcal{N}_R$ and $\mathcal{N}_{R^\circ}$ coincide, where $\mathcal{N}_{R^\circ}$ is the nullity distribution of the h-curvature $\overset{\circ}{R}$ of Berwald connection.*

Proof. Let (M,E) be a Landesberg space. Then, the hv-curvature P of Cartan connection vanishes and thus $\mathcal{N}_P = H(TM)$. Consequently, $\mathcal{C}' = 0$, by Proposition 5.2 **(c)**. Hence, by Lemma 3.5 **(a)**, we get

$$R(X,Y)Z = \overset{\circ}{R}(X,Y)Z + \mathcal{C}(F\mathfrak{R}(X,Y),Z).$$

Let $X \in \mathcal{N}_R$, then $X \in \mathcal{N}_{\mathfrak{R}}$ and thus $\mathfrak{R}(X,Y) = 0$, hence, $X \in \mathcal{N}_{R^\circ}$. Consequently, $\mathcal{N}_R \subseteq \mathcal{N}_{R^\circ}$. Conversely, let $X \in \mathcal{N}_{R^\circ}$, then $X \in \mathcal{N}_{\mathfrak{R}}$ [15] and thus $\mathfrak{R}(X,Y) = 0$, hence, $X \in \mathcal{N}_R$. Consequently, $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_R$. □

The nullity distribution $\mathcal{N}_P$ is not in general completely integrable as shown by the following example.

Example 5.4. Let $M = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 : x_2 \neq 0\}$,
$U = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_2 \neq 0;\ y_1, y_2 \neq 0\} \subset TM$.
Let the energy function E be defined on U by: $E = e^{-x_1}(e^{-x_1x_3}y_1^2y_3 + x_2y_2^3)^{2/3}$.

For simplicity, let $\sigma_1 := e^{-x_1x_3}y_1^2y_3 + x_2y_2^3$, $\sigma_2 := 7e^{-x_1x_3}y_1^2y_3 + 12x_2y_2^3$ and $\sigma_3 := e^{x_1x_3}(5e^{-x_1x_3}y_1^2y_3 + 3x_2y_2^3)$. Then, the non-vanishing components P^h_{ijk} of the hv-curvature tensor P are:

$$P^1_{111} = \frac{-3x_2y_2^3}{32y_1\sigma_1}, \quad P^2_{111} = \frac{-y_2\sigma_2}{32y_1^2\sigma_1}, \quad P^3_{111} = \frac{-9x_2y_2^3y_3}{32y_1^2\sigma_1}, \quad P^1_{112} = \frac{3x_2y_2^2}{32\sigma_1} = P^1_{121},$$

$$P^2_{112} = \frac{\sigma_2}{32y_1\sigma_1} = P^2_{121}, \quad P^3_{112} = \frac{9x_2y_2^2y_3}{32y_1\sigma_1} = P^3_{121}, \quad P^1_{122} = \frac{-3x_2y_1y_2}{32\sigma_1},$$

$$P^2_{122} = \frac{-\sigma_2}{32y_2\sigma_1}, \quad P^3_{122} = \frac{-x_2y_2y_3}{32\sigma_1}, \quad P^1_{211} = \frac{3x_2y_2^2}{32\sigma_1}, \quad P^2_{211} = \frac{x_2y_2^3}{16y_1\sigma_1},$$

$$P^3_{211} = \frac{3x_2y_2^2\sigma_3}{16y_1^3\sigma_1}, \quad P^1_{221} = \frac{-3x_2y_1y_2}{32\sigma_1} = P^1_{212}, \quad P^2_{221} = \frac{-3x_2y_2^2}{16\sigma_1} = P^2_{212},$$

$$P^3_{221} = \frac{-3x_2y_2}{16y_1^2\sigma_1} = P^3_{212}, \quad P^1_{222} = \frac{3x_2y_1^2}{32\sigma_1}, \quad P^2_{222} = \frac{3x_2y_1y_2}{16\sigma_1}, \quad P^3_{222} = \frac{3x_2\sigma_3}{16\sigma_1},$$

$$P^2_{311} = \frac{y_1y_2e^{-x_1x_3}}{32\sigma_1}, \quad P^3_{311} = \frac{-3x_2y_2^3}{32y_1\sigma_1}, \quad P^2_{312} = \frac{-x_2y_1^2e^{-x_1x_3}}{32y_1^3\sigma_1} = P^2_{321},$$

$$P^3_{312} = \frac{3x_2y_2^2}{32\sigma_1} = P^3_{321}, \qquad P^2_{322} = \frac{y_1^3 e^{-x_1x_3}}{32y_2\sigma_1}, \qquad P^3_{322} = \frac{-3x_2y_1y_2}{32\sigma_1}.$$

Now, let $X \in \mathcal{N}_P$. The equation $P(X,Y)Z = 0, \forall\, Y, Z \in H(TM)$, is written locally in the form

$$X^j P^h_{ijk} = 0.$$

This yields the system of equations

$$y_2X^1 - y_1X^2 = 0.$$

Thus, the solution of the above system is $X^1 = t_1, X^2 = \frac{y_2}{y_1}t_1$ and $X^3 = t_3$, $t_1, t_3 \in \mathbb{R}$. Hence, $X = t_1(h_1 + \frac{y_2}{y_1}h_2) + t_3h_3$ and $\mu_P = 2$. Now, let $X, Y \in \mathcal{N}_P$ be such that $X = h_1 + \frac{y_2}{y_1}h_2$ and $Y = h_3$. By simple calculations, the bracket $[X, Y] = [h_1 + \frac{y_2}{y_1}h_2, h_3] = -\frac{1}{2}y_1\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_3}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_P$ is not complectly integrable. □

Nevertheless, we have

Theorem 5.5. *Let μ_P be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_P$ is completely integrable on U if and only if, for all $X, Y \in \mathcal{N}_P$, $\mathfrak{R}(X,Y) = 0$ and $(D_{JZ}R)(X,Y) = R(Y, F\mathcal{C}(X,Z)) - R(X, F\mathcal{C}(Y,Z))$.*

Proof. Let $X, Y \in \mathcal{N}_P$. Then, $\mathfrak{R}(X,Y) = 0$ and $(D_{JZ}R)(X,Y) = R(Y, F\mathcal{C}(X,Z)) - R(X, F\mathcal{C}(Y,Z))$, $\forall Z \in \mathfrak{X}(TM)$. As $\mathfrak{R}(X,Y) = 0$, then the bracket $[hX, hY]$ is horizontal. Making use of Lemma 3.7 **(f)** and Lemma 4.1 **(c)**, we get

$$\begin{aligned}(D_{hX}P)(Y,Z) - (D_{hY}P)(X,Z) = 0 &\Longrightarrow P(D_XY - D_YX, Z) = 0\\ &\Longrightarrow P([X,Y] + \mathfrak{R}(X,Y), Z) = 0\\ &\Longrightarrow P([X,Y], Z) = 0\\ &\Longrightarrow [X,Y] \in \mathcal{N}_P.\end{aligned}$$

Hence $\mathcal{N}_P$ be completely integrable.

Conversely, let $\mathcal{N}_P$ be completely integrable. Then, if $X, Y \in \mathcal{N}_P$, the bracket $[hX, hY]$ is horizontal, thus, $\mathfrak{R}(X,Y) = 0$. Also, by Lemma 3.7 **(f)** and the fact $P([hX, hY], Z) = (D_{hX}P)(Y,Z) - (D_{hY}P)(X,Z) = 0$, we have $(D_{JZ}R)(X,Y) = R(Y, F\mathcal{C}(X,Z)) - R(X, F\mathcal{C}(Y,Z))$, $\forall X, Y \in \mathcal{N}_P$, $\forall Z \in \mathfrak{X}(TM)$. □

Remark 5.6. The class of Finsler spaces with vanishing h-curvature satisfy the conditions of Theorem 5.5. Consequently, for such spaces, $\mathcal{N}_P$ is completely integrable.

Moreover, we have

Proposition 5.7. *A sufficient condition for $\mathcal{N}_P$ to be completely integrable is that*

$$\mathcal{N}_P \subset N_R.$$

Proof. Let $\mathcal{N}_P \subset N_R$ and $X, Y \in \mathcal{N}_P$, $Z \in \mathfrak{X}(TM)$. Then, $X, Y \in \mathcal{N}_R$ and hence $X, Y \in \mathcal{N}_{\mathfrak{R}}$, consequently, $\mathfrak{R}(X,Y) = 0$. Also, by Lemma 3.7 **(f)**, we have $(D_{JZ}R)(X,Y) = R(Y, F\mathcal{C}(X,Z)) - R(X, F\mathcal{C}(Y,Z))$, $\forall X, Y \in \mathcal{N}_P$, $\forall Z \in \mathfrak{X}(TM)$. Hence, by Theorem 5.5, $\mathcal{N}_P$ is completely integrable. □

6. Nullity distribution of Cartan v-curvature

In this section, we study the nullity distribution of the v-curvature Q of Cartan connection. The nullity distribution of Q is defined in a similar manner as that of R (Definition 4.3)

Proposition 6.1. *The nullity distribution of Q satisfies:*

(a) $\mathcal{N}_Q \neq \phi$.

(b) $S \in \mathcal{N}_Q$.

(c) *If $Z \in \mathcal{N}_Q$, then $Q(X,Y)Z = 0, \forall X, Y \in \mathfrak{X}(TM)$.*
That is, $Q(X,Y)Z$ vanishes whenever X, Y or Z is a Q-nullity vector field.

(d) *If $X, Y \in \mathcal{N}_Q(z)$, then $F[JX, JY] \in \mathcal{N}_Q$.*

Proof.
(b) Follows from the fact that $Q(S,X)Y = 0$. (Lemma 3.6 **(d)**)
(c) Follows from Lemma 3.7 **(b)**.
(d) Let $X, Y \in \mathcal{N}_Q$, then Propositions 6.1 and Lemma 3.7 **(h)** lead to

$$\begin{aligned}0 &= (D_{JX}Q)(Y,Z) + (D_{JY}Q)(Z,X) + (D_{JZ}Q)(X,Y)\\ &= -Q(D_{JX}Y, Z) - Q(Z, D_{JY}X)\\ &= Q(D_{JY}X - D_{JX}Y, Z)\\ &= Q(F[JX,JY], Z).\end{aligned}$$

Since $[JX, JY]$ is vertical and $FJ = h$, hence $F[JX, JY]$ is horizontal. Consequently, $F[JX, JY] \in \mathcal{N}_Q$. □

The nullity distribution $\mathcal{N}_Q$ is not in general completely integrable as shown by the following example.

Let $E = x_4 y_1 (y_2^3 + y_3^3 + y_4^3)^{1/3}$. Then, we have:

Example 6.2. $M = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 \neq 0\}$,
$U = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x_2 \neq 0;\ y_1, y_3, y_4 \neq 0\} \subset TM$.
Let the energy function E be defined on U by $E = x_2 y_1^2 e^{-y_3/y_4} + y_2^2$.

Then, the independent non-vanishing components of the v-curvature Q^h_{ijk} of Cartan connection are:

$$Q^3_{113} = \frac{-y_3}{2y_1^2 y_4}, \qquad Q^4_{113} = \frac{-1}{2y_1^2}, \qquad Q^3_{114} = \frac{y_3^2}{2y_1^2 y_4^2}, \qquad Q^4_{114} = \frac{y_3}{2y_1^2 y_4},$$

$$Q^3_{134} = \frac{-y_3}{2y_1 y_4^2}, \qquad Q^4_{134} = \frac{-1}{2y_1 y_4}, \qquad Q^3_{313} = \frac{-1}{2y_1 y_4}, \qquad Q^1_{314} = \frac{y_3}{4y_4^3},$$

$$Q^1_{331} = \frac{-1}{4y_4^2}, \qquad Q^1_{334} = \frac{-y_1}{4y_4^3}, \qquad Q^3_{334} = \frac{-1}{2y_1^2}, \qquad Q^3_{341} = \frac{-y_3}{2y_1^2 y_4},$$

$$Q^1_{413} = \frac{y_3}{4y_4^3}, \qquad , Q^3_{413} = \frac{y_3}{y_1 y_4^2}, \qquad Q^1_{414} = \frac{-y_3^2}{4y_4^4}, \qquad Q^3_{414} = \frac{-y_3}{y_1 y_4^2},$$

$$Q^4_{414} = \frac{-y_3}{2y_1y_4^2}, \quad Q^4_{413} = \frac{1}{2y_1^2y_4}, \quad Q^4_{423} = \frac{1}{2y_1y_4}, \quad Q^1_{434} = \frac{y_1y_3}{4y_4^4}, \quad Q^3_{434} = \frac{y_3}{y_4^3}.$$

Now, let $X \in \mathcal{N}_Q$, the equation $Q(X,Y)Z = 0$, $\forall Y, Z \in H(TM)$, is written locally in the form

$$X^j Q^h_{ijk} = 0.$$

This is equivalent to the system of equations:

$$y_4X^3 - y_3X^4 = 0,$$

$$y_4X^1 - y_1X^4 = 0,$$

$$y_3X^1 - y_1X^3 = 0.$$

From the above system, we have $X^2 = t$, $X^4 = t'$, $X^1 = \frac{y_1}{y_4}t'$, $X^3 = \frac{y_3}{y_4}t'$, $t, t' \in \mathbb{R}$. Hence, $X = th_2 + t'(\frac{y_1}{y_4}h_1 + \frac{y_3}{y_4}h_3 + h_4)$ and $\mu_Q = 2$. Now, let $X, Y \in \mathcal{N}_Q$ be such that $X = h_2$ and $Y = \frac{y_1}{y_4}h_1 + \frac{y_3}{y_4}h_3 + h_4$. Then, the bracket $[X,Y] = [h_2, \frac{y_1}{y_4}h_1 + \frac{y_3}{y_4}h_3 + h_4] = -\frac{y_1y_2}{2x_2^2y_4}\frac{\partial}{\partial y_1} + \frac{y_1^2(5y_3-2y_4)}{4x_2y_4^2}e^{-y_3/y_4}\frac{\partial}{\partial y_2} + \frac{y_4}{2x_2^2}\frac{\partial}{\partial y_4}$, which is vertical. Consequently, the nullity distribution $\mathcal{N}_Q$ is not complectly integrable. □

Nevertheless, we have

Theorem 6.3. *Let μ_Q be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_Q$ is completely integrable on U if and only if, for all $X, Y \in \mathcal{N}_Q$, $\mathfrak{R}(X,Y) = 0$ and the tensor*

$$A(X,Y,Z) := P(F\mathcal{C}(Z,X),Y) - (D_{JX}P)(Y,Z) - (D_{JZ}P)(X,Y), \forall Z \in \mathfrak{X}(TM)$$

is symmetric in X and Y.

Proof. Let $X, Y \in \mathcal{N}_Q$. Then, $\mathfrak{R}(X,Y) = 0$ and the tensor $A(X,Y,Z)$ is symmetric in the first two arguments. By Lemma 3.7 **(g)**, we have

$$\begin{aligned}(D_{hX}Q)(Y,Z) = (D_{JY}P)(X,Z) - (D_{JZ}P)(X,Y) + P(F\mathcal{C}(X,Y),Z) \\ -P(F\mathcal{C}(Z,X),Y) - Q(F\mathcal{C}'(X,Y),Z).\end{aligned} \tag{6.1}$$

Interchange X with Y in the above equation, we get

$$\begin{aligned}(D_{hY}Q)(X,Z) = (D_{JX}P)(Y,Z) - (D_{JZ}P)(Y,X) + P(F\mathcal{C}(Y,X),Z) \\ -P(F\mathcal{C}(Z,Y),X) - Q(F\mathcal{C}'(Y,X),Z).\end{aligned} \tag{6.2}$$

Making use of the symmetry of $\mathcal{C}$ and $\mathcal{C}'$, Equations (6.1) and (6.2) give

$$(D_{hX}Q)(Y,Z) - (D_{hY}Q)(X,Z) = A(X,Y,Z) - A(Y,X,Z). \tag{6.3}$$

Then, by the symmetry of $A(X,Y,Z)$ in X, Y, we get

$$Q(D_{hY}X - D_{hX}Y, Z) = 0.$$

Consequently, it follows from Lemma 4.1 that

$$Q([X,Y] + \mathfrak{R}(X,Y), Z) = 0.$$

Since $\mathfrak{R}(X,Y) = 0$, $[hX, hY] = [X,Y]$ is horizontal and so $[X,Y] \in \mathcal{N}_Q$. Consequently, $\mathcal{N}_Q(z)$ is completely integrable.

Conversely, let $\mathcal{N}_Q$ be completely integrable. Then, for all $X, Y \in \mathcal{N}_Q$, the bracket $[hX, hY] \in \mathcal{N}_Q$, i.e., $[hX, hY]$ is horizontal and hence $\mathfrak{R}(X,Y) = 0$. Moreover, by (6.3) and the fact that $Q([hX, hY], Z) = 0$, the tensor $A(X,Y,Z)$ is symmetric in X and Y. □

Remark 6.4. The class of Minkoweski spaces satisfies the conditions of the above theorem. Consequently, a Minkoweski space has a completely integrable $\mathcal{N}_Q$.

Definition 6.5. *Let (M,E) be a Finsler manifold. The angular metric $\hbar$ on TM is defined by*

$$\hbar(X,Y) = g(X,Y) - \ell(X)\ell(Y),$$

where g is the metric tensor on TM given by (3.2) and $\ell(X) := \frac{1}{\sqrt{2E}} g(X,C)$.

It should be noted that the trace of $\hbar$ is $(2n-1)$.

Definition 6.6. *A Finsler space (M,E) of $\dim \geq 4$ is said to be S_3-like if*

$$Q(X,Y,Z,W) = r\{\hbar(JX,JZ)\hbar(JY,JW) - \hbar(JX,JW)\hbar(JY,JZ)\},$$

where $Q(X,Y,Z,W) = g(Q(X,Y)Z, JW)$ and r is a scalar function.

Theorem 6.7. *Let (M,E) be an S_3-like space. Then, the index of nullity μ_Q takes its maximal value.*

Proof. Let (M,E) be an S_3-like space and $X \in \mathcal{N}_Q$, then we have

$$r\{\hbar(JX,JZ)\hbar(JY,JW) - \hbar(JX,JW)\hbar(JY,JZ)\} = 0.$$

Taking the trace with respect to JX and JZ , we get

$$(2n-2)r\hbar(JY,JW) = 0.$$

Again, taking the trace of the above equation, we have

$$(2n-1)(n-1)r = 0.$$

As $n \geq 4$, then $r = 0$ and consequently $Q = 0$. □

References

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler isotropes*, C. R. Acad. Sci. Paris, Sér. A, **252** (1961), 2061–2063.

[2] H. Akbar-Zadeh, *Espaces de nullité de cértains operateurs en géométrie des sous-variétés*, C. R. Acad. Sci. Paris, Sér. A, **274** (1972), 490-493.

[3] H. Akbar-Zadeh, *Espaces de nullité en géométrie Finslérienne*, Tensor, N. S., **26** (1972), 89-101.

[4] B. Bidabad and M. Refie-Rad, *On the k-nullity foliations in Finsler geometry and completeness*, Bull. Iranian Math. Soc., **37**, **4** (2011), 1-18. Arxiv: 1101.1496v1 [math. DG].

[5] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Paris, Gauthier-Villars, 1928; 2nd. Ed., 1946.

[6] S. S. Chern and N. H. Kuiper, *On some theorems on isometric imbedding of compact Riemann manifolds in Euclidean space*, Ann. Math., **56** (1952), 313-316.

[7] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, I, Ann. Proc. Kon. Ned. Akad., A, **59** (1956), 338–359.

[8] J. Grifone, *Structure presque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, **22, 1** (1972), 287–334.

[9] J. Grifone, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier, Grenoble, **22, 3** (1972), 291–338.

[10] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18, 1** (1968), 241–260.

[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, II, Interscience, New York, 1969.

[12] R. Maltz, *The nullity space of the curvature operator*, Cahiers Top. Géom. Diff., **8** (1965), 1–20.

[13] J. Szilasi and Cs. vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paedagog. Nyházi, N. S., **16** (2000), 33–63.

[14] Nabil L. Youssef, *Distribution de nullité du tensor de courbure d'une connexion*, C. R. Acad. Sci. Paris, Sér. A, **290** (1980), 653–656.

[15] Nabil L. Youssef, *Sur les tenseurs de courbure de la connexion de Berwald et ses distributions de nullité.* Tensor, N. S., **36** (1982), 275-280.

[16] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., **48**, **4** (2008), 857–893. ArXiv: 0704.0053 [math. DG].